A particle orbits the origin on a circle of radius $a$ in the plane perpendicular to $\hat{v} = \frac{1}{\sqrt{2}}(1,1,0)$ defined in an $(x,y,z)$ coordinate frame.

a. Write the equations of motion in terms of 2 suitable Cartesian coordinates $(u,z)$.

The coordinate frame in which the particle orbits is $(u,z)$ where $\hat{u} = \hat{z} \times \hat{v} = \frac{1}{\sqrt{2}}(-1,1,0)$. The equations of motion for a circular orbit are

\[ \ddot{u} = -\omega^2 u \quad \ddot{z} = -\omega^2 z \quad u^2 + z^2 = a^2 \]

One particular solution is

\[ \vec{r} = \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a \cos \omega t \\ 0 \\ a \sin \omega t \end{pmatrix} \]

b. Devise a rotation matrix that transforms the $(u,z)$ plane to the $(x,y,z)$ plane.

\[ R = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

c. Apply the transformation to obtain the equations of motion in $(x,y,z)$.

\[ \vec{r}' = R \vec{r} = \frac{a}{\sqrt{2}} \begin{pmatrix} -\cos \omega t \\ \cos \omega t \\ \sqrt{2} \sin \omega t \end{pmatrix} \]
2. A projectile is fired from a gun at an angle $\alpha$ above horizontal. What is the maximum value of $\alpha$ for which the distance from the gun to the projectile is always increasing (until the projectile hits the ground of course)? Make a clear figure and explain clearly what is asked, and show your work neatly.

The projectile motion is the familiar parabola:

\[ x = v_1 t \]
\[ y = v_2 t - \frac{1}{2} gt^2 \]

\[ v_1 = v_0 \cos \alpha \quad v_2 = v_0 \sin \alpha \]

The distance from gun to projectile is

\[ r = \sqrt{x^2 + y^2} = \sqrt{v_0^2 t^2 - g v_2 t^3 + \frac{1}{4} g^2 t^4} \]

We want to require that this distance does not have an extremum:

\[ \frac{dr}{dt} = \frac{2v_0^2 t - 3g v_2 t^2 + g^2 t^3}{2\sqrt{v_0^2 t^2 - g v_2 t^3 + \frac{1}{4} g^2 t^4}} = 0 \]

\[ t = \frac{3v_2 \pm \sqrt{9v_2^2 - 8v_0^2}}{2g} \]

There is no extremum iff the radicand is negative:

\[ \tan \alpha < \sqrt{8} \]

\[ \alpha < 71^\circ \]
3. Consider a damped harmonic oscillator satisfying the equation of motion
\[ \ddot{x} + 2\beta \dot{x} + \omega_0^2 x = a(t) \]
a. Solve for \( x(t) \) if there is no driving term (\( a(t) = 0 \)) and the oscillator has initial conditions
\[ x(0) = a \quad \dot{x}(0) = 0 \]

Note: Prof. McIntyre goofed on this one. In my mind was the case of an underdamped oscillator, but I did not stipulate \( \beta < \omega_0 \). The solution below is for this underdamped case. If you worked the problem for the more general case and did not receive full credit, please see me. PM

Obtain a solution using the ansatz \( x = \text{Re}(x_0 e^{i(\omega t + \phi)} \cdot e^{-\beta t}) \)
\[
(i \omega_1 - \gamma)^2 + 2\beta (i \omega_1 - \gamma) + \omega_0^2 = 0
\]
\[ \gamma = \beta \]
\[ \omega_1 = \sqrt{\omega_0^2 - \beta^2} \]

Now apply the initial conditions:
\[ \text{Re}(x_0 e^{i\phi}) = a \quad x_0 = a \sec \phi \]
\[ \text{Re}[x_0 (i \omega_1 - \beta) e^{i\phi}] = 0 \]
\[ i\omega \cos \phi - \beta \cos \phi - \omega \sin \phi - i\beta \sin \phi \]
\[ \tan \phi = \frac{-\beta}{\omega_1} \quad \sec \phi = \sqrt{1 + \tan^2 \phi} = \frac{\omega_0}{\omega_1} \]
\[
x(t) = \frac{a \omega_0}{\omega_1} e^{-\beta t} \cos(\omega_1 t + \phi)
\]
\[
x(t) = a e^{-\beta t} \left[ \cos \omega_1 t + i \frac{\beta}{\omega_1} \sin \omega_1 t \right]
\]

Note: check that the initial conditions are in fact met with the solution!

b. Now suppose there is a driving term that is a square pulse of constant amplitude \( a_0 \) for times \( 0 < t < T \), 0 at all other times. Use Green’s method to calculate the response \( x(T) \).
\[
x(T) = \frac{a_0}{\omega_1} \int_0^T e^{-\beta(T-t)} \sin(\omega_1 (T-t)) \, dt'
\]
\[
= \frac{a_0}{2i\omega_1} e^{-\beta T} \left[ e^{i\omega_1 T} \int_0^T e^{(-i\omega_1 + \beta)t'} \, dt' - e^{-i\omega_1 T} \int_0^T e^{(i\omega_1 + \beta)t'} \, dt' \right]
\]
\[
= \frac{a_0}{2i\omega_1} \left[ \frac{1 - e^{i\omega_1 T - \beta T}}{-i\omega_1 + \beta} - \frac{1 - e^{-i\omega_1 T - \beta T}}{i\omega_1 + \beta} \right] = \frac{a_0}{\omega_1^2} \left[ 1 - e^{-\beta T} \cos \omega_1 T \right] \left[ \frac{\beta}{\omega_1} \sin \omega_1 T \right]
\]

Check the limit \( T \to 0 \): \( x(T) \to 0 \) as it should.
Note: \( a_0 \) has units of acceleration.