Galilean Relativity

Galileo Galilei laid the framework for our description of the relative motion of bodies in classical mechanics. Galilean relativity is rooted in the invariance of the laws of physics under a translation in space. We can describe the position of the airplane in Figure 1 in terms of coordinates \( \mathbf{r}' = (x', y', z') \) in the coordinate frame \( S' \), and equivalently in terms of the coordinates \( \mathbf{r} = (x, y, z) \) in the coordinate frame \( S \). If \( S' \) is displaced from \( S \) by a vector displacement \( \mathbf{R} \), then translational invariance requires that

\[
\mathbf{r} = \mathbf{R} + \mathbf{r}' \tag{1}
\]

This description of positions measured in the different systems leads directly into the relation of velocities measured in different systems, by simply taking the time derivative of Eq. 1:

\[
\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{r}'}{dt}, \quad \mathbf{v} = \mathbf{V} + \mathbf{v}' \tag{2}
\]

Two frames of reference that are moving with respect to one another with a constant velocity \( \mathbf{v} \) are called inertial frames.

*Velocity transforms as a simple translation when we transform between inertial frames.*

Now consider the displacement between two points on a bar AB, as observed in the two inertial frames of Figure 2. Suppose that frame \( S' \) is moving with respect to frame \( S \) with a velocity \( \mathbf{V} = \dot{\mathbf{V}} \).
Then the length $L$ of the bar in the blue frame can be related to the length $L'$ observed in the rest frame:

$$L = \vec{r}_b - \vec{r}_A = (\vec{R} + \vec{r}_B') - (\vec{R} + \vec{r}_A') = \vec{r}_b' - \vec{r}_A' = L'$$

In Galilean transformations, the length of an extended object is invariant under transformations among inertial frames.
Special Relativity

We will develop the ability to describe the relative motion of objects even in the limit that they move at or near the speed of light. Light represents a very special case in the description of motion, because light can travel through entirely empty space, always with the same speed (magnitude of velocity) $c = 3 \times 10^{10} \text{ m/s}$. Other forms of wave motion – sound and water waves, for example – must travel as an excitation of a background medium. If you remove the medium, the wave cannot propagate. Yet light travels from the most distant stars through empty space!

Lorentz and Maxwell developed an understanding of this wave motion when they unified the description of electric and magnetic fields to obtain the solutions for an electromagnetic wave. The wave propagates as a packet of electric field $\vec{E}$ and magnetic field $\vec{B}$, in which each field oscillates in synchronism and the packet moves in the direction $\hat{k} = \vec{E} \times \vec{B}$. The packet propagates at the velocity $c$ in the absence of any medium.

This notion of propagation through empty space troubled the physicists of 1900 so much, that they hypothesized that perhaps even empty space contains a background medium, termed at the time the “luminiferous ether”, that supports the propagation of light. There was no evidence whether such a medium existed or not, so Albert Michelson set out to try to measure it.

Michelson used an interferometer to compare the speed of light in two perpendicular directions. We will discuss the details of his experiment in a future lecture. By comparing the number of wavelengths by which light travels in the x- and y- paths of Figure 3, he was able to measure the velocity of light in each direction. Now suppose that the apparatus is moving through the “ether”, in which light propagates, with a velocity $\vec{v}$. We showed above that the path length is invariant in Galilean transformation, but the velocity of light should change. Therefore the number of wavelengths $\lambda$ in each path should change according to its orientation with respect to the motion of the apparatus through the ether.

The result of the experiment is that the relative number of wavelengths in the two paths remains unchanged, no matter how the paths are oriented with respect to North/South, the rotation of Earth, and the time of year in Earth’s revolution around the sun. Since the orientation with respect to the motion through any “ether” must change with all three of these orientations, we must conclude that the velocity of light is constant in all inertial frames.

![Figure 3. Michelson-Morley experiment to measure the speed of light.](image-url)
This result, obtained in 1887, launched a revolution in physics. It required a wholly new way of describing relative motion. The new way must yield the Galilean transformation for velocities $v<<c$, but it must yield that the velocity of light is the same in all inertial frames.

In order to appreciate how Einstein devised his Theory of Special Relativity, consider again the Galilean transformation treats space and time in a transformation from a frame $S'$ to a frame $S$ that is moving with a constant velocity $v = \hat{x}:$

$$x = X + x' = vt + x'$$
$$y = y'$$
$$z = z'$$
$$t = t'$$

Einstein was well-versed in mathematics, and in the way that coordinate systems transform under linear transformations. He recognized that the above transformation, if viewed as a transformation in a 4-dimensional space, made no sense. It mixed the space coordinate $x'$ and $t'$ to obtain $x$, yet it left $t'$ unchanged!

Einstein realized that a mixing of $x'$ and $t'$ corresponds to a rotation between those coordinates, analogous to a rotation by an angle $\theta$ between two spatial coordinates:

$$x = x'\cos \theta + y'\sin \theta$$
$$y = y'\cos \theta - x'\sin \theta$$

He reasoned that perhaps the correct expression for transformation between inertial frames was such a rotation. He sought one that would yield the further result that the speed of light is the same in all frames. He found that there is a unique transformation that does it. If the frame $S$ is moving with velocity $\vec{v} = v\hat{x}$ with respect to the frame $S'$, then

$$x = \gamma(x' + \beta ct')$$
$$y = y'$$
$$z = z'$$
$$ct = \gamma(ct' + \beta c')$$
$$\beta = v/c$$
$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

This transformation is called the Lorentz transformation. We will evaluate the velocity $\frac{dx}{dt}$ and show that when $\frac{dx'}{dt'} = c$, $\frac{dx}{dt} = c$ also. We will also show that the Lorentz transformation reduces to the Galilean transformation when $\beta << 1$.

Note that the space coordinates $y, z$ that are transverse to the relative motion do not change under the Lorentz transformation. They are Lorentz invariants. We will see that using such invariants to best advantage makes a great simplification of most problems in relativity.