The Lorentz transformation yields the limiting cases of the Galilean transformation for $v \ll c$, and that $c$ is the same in all frames:

\[
\begin{align*}
x &= \gamma (x' + \beta ct') \\
y &= y' \\
z &= z' \\
c t &= \gamma (c t' + \beta x')
\end{align*}
\]

$\beta = v / c$

\[
\gamma = \frac{1}{\sqrt{1 - \beta^2}}
\]

We will now evaluate velocity, and show that both limiting cases are obtained.
\[ v = \frac{dx}{dt} \]
\[ dx = \gamma(dx' + \beta c dt') \]
\[ cdt = \gamma(cdt' + \beta dx') \]
\[ \frac{dx}{dt} = \frac{\gamma(dx' + \beta c dt')} \gamma(cdt' + \beta dx') = \frac{dx'}{dt'} + \beta c \]
\[ 1 + \frac{\beta}{c} \frac{dx'}{dt'} \]
\[ v = \frac{v' + \beta c}{1 + \frac{\beta c}{v'}} \]

Now evaluate the 2 cases: \( v \ll c \) and \( v \rightarrow c \)
Case 1: \( v \ll c : \beta \ll 1 \)  
Recover Galilean transformation!

\[
\frac{dx}{dt} \approx \frac{dx'}{dt'} + v
\]

Case 2: \( \frac{dx'}{dt'} \rightarrow c : \)  
Speed of light is the same in all frames!

\[
\frac{dx}{dt} = \frac{(1 + \beta)c}{(1 + \beta)} = c
\]

The Lorentz transformation is unique in yielding these two vital limiting cases. It preserves classical mechanics in the limit when all velocities are slow compared to that of light, and at the same time it yields the speed of light is constant in all frames, as is observed in the Michelson-Morley experiment.
It is useful to work with Lorentz invariants when analyzing problems in special relativity. We have several:

Assume as before that the relative motion of the two frames is \( v = \beta c \hat{x} \)

Then

\[
\begin{align*}
  y &= y' \\
  z &= z' \\
  s^2 &\equiv x^2 + y^2 + z^2 - c^2 t^2
\end{align*}
\]

and one more:

Let's prove that \( s^2 \) is an invariant:

\[
\begin{align*}
  s^2 &= \gamma^2 (x' + \beta ct')^2 + y'^2 + z'^2 - \gamma^2 (ct' + \beta x')^2 \\
  &= y'^2 + z'^2 + x'^2 \gamma^2 \left[1 - \beta^2\right] + c^2 t'^2 \gamma^2 \left[\beta^2 - 1\right] + 2 \beta \gamma^2 c x' t' - 2 \beta \gamma^2 x' t' \\
  &= x'^2 + y'^2 + z'^2 - c^2 t'^2 \\
  &\quad QED
\end{align*}
\]
Fitzgerald contraction, Time dilation

Now consider a rod, oriented in the $\hat{x}$ direction. We measure its length $L'$ in its rest frame.

Now let’s find out what is its length as seen in a moving frame, moving at velocity $v$:

Solve for $x$, $t$ in the Lorentz transformation:

$$x = \gamma (x' - \beta ct')$$

$$ct = \gamma (ct' - \beta x')$$
To measure the length \( L = x_2 - x_1 \) of the rod, we will measure the positions of both ends at the same time: \( t_2 = t_1 \).

\[
L' = x_2' - x_1' = \gamma (x_2 + \beta ct_2) - \gamma (x_1 + \beta ct_1) \\
= \gamma (x_2 - x_1) \\
= \gamma L
\]

The length of the rod in its rest frame (\( L' \)) is defined as the \textit{proper length}. \textit{The proper length is always the longest apparent length of an object!} The object appears shorter in any moving frame – Fitzgerald contraction.
Time dilatation

Now observe a clock that is located at rest with respect to the rod. We will use the clock to measure a time interval
\[ T' = t_2' - t_1' \]

The clock does not move in the rest frame, so \( x_1' = x_2' \)

Now measure the time interval in the moving frame:
\[
cT = \gamma (\beta ct_2' - x_2') - \gamma (\beta ct_1' - x_1') \\
= c\gamma (t_2' - t_1') = \gamma cT'
\]

The time interval observed in the rest frame of the clock is always shorter than that observed in any moving frame. It is called the **proper time.**
Transformation of angles

Suppose that, instead of orienting the rod in the direction of relative motion of the two frames, it makes angle $\vartheta'$ in the rest frame:

\[
L'_x = L' \cos \vartheta' \quad \text{and} \quad L'_y = L' \sin \vartheta'.
\]
In the moving frame, the x-component appears shorter:

\[ L_y = L'_{y} = L \sin \vartheta \]

\[ L_x = \frac{1}{\gamma} L'_{x} = L \cos \vartheta \]

so \[ \tan \vartheta = \frac{L \sin \vartheta}{L \cos \vartheta} = \frac{L_y}{L_x} = \frac{\gamma L'_{y}}{L'_{x}} = \frac{\gamma \sin \vartheta'}{\cos \vartheta'} = \gamma \tan \vartheta' \]

\[ \tan \vartheta = \gamma \tan \vartheta' \]
Twin paradox

Suppose two twins are NASA astronauts in some distant time, when star travel is possible. One twin boards a space ship, accelerates to $v=0.99c$ quickly, and then travels to a star that is 10 light years away. After she arrives, she turns around and travels home at the same speed. The other twin stays at home.

How much time has elapsed in each twin’s life during the time of the voyage?
Define the frame $S'$ to be the rest frame of the voyager, and $S$ to be the rest frame of the twin that stayed on Earth.

At the end of the trip out, the elapsed times in each frame were

$$T_1' = T_1 / \gamma = \frac{L_1}{\gamma \beta c} = \frac{10 \text{ years}}{\gamma}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} = 7$$

so $T_1' = 1.5 \text{ years}$

The elapsed time for the return trip should be the same as on the trip out for each twin.
So finally, when the traveling twin emerges from her ship on Earth, she has aged by 3 years, while her sister has aged by 20 years!

The paradox is that, both before she left and after her return, the two twins are in the same frame! So how can their clocks not agree?

The resolution of this paradox illustrates a limitation of special relativity: it can only describe inertial frames of reference. When the traveling twin accelerated to start, stop, return, and stop again, the acceleration breaks the synchronism of the clocks over the time interval of the trip. This paradox was one of the puzzles that ultimately led Einstein to develop the Theory of General Relativity.