Problem-solving skills

In most problems, you are given information about two points in space-time, and you are asked to find information about the space or time separation in another frame of reference.

The single most universal way to attack such problems is to write down the 4-vector points in the one frame, and then transform them to the other frame to obtain the desired information.
Example: problem 5.14

For an observer in a rest frame S, an explosion occurs at \(x_1=0, t_1=0\). A second explosion occurs at \(x_2=500 \text{ m} \) at time \(t_2=10^{-6} \text{ s}\). Calculate the velocity of a second observer if the second observer is to observe the two flashes simultaneously.

• *Simultaneous*: in the second observer’s frame, \(t_1' = t_2'\)

• Write down the 4-vectors in frame S:

\[
\begin{align*}
  s_1 &= (x_1, y_1, z_1, ct_1) = (0,0,0,0) \\
  s_2 &= (500m,0,0,c \cdot 10^{-6} \text{ s}) = (500,0,0,300) \text{m}
\end{align*}
\]
• Construct the Lorentz transform to the other frame:

\[ x' = \gamma(x - \beta ct) \quad \text{ct}' = \gamma(ct - \beta x) \]

Check the sign of the second terms: look at how the origin of the frame S (x=0) moves as seen in the frame S’: x decreases, so the second terms need the - sign. (It depends upon the statement of the problem and which way you assume the relative velocity to point.)

\[ s_1' = (0,0,0,0) \quad s_2' = \gamma(500 - 300\beta,0,0,300 - 500\beta) \]

\[ t_2' - t_1' = \gamma(300 - 500\beta) = 0 \quad \Delta t = 0 \text{ for simultaneity.} \]

\[ \beta = 0.6 \quad v = 1.8 \times 10^8 \text{ m/s} \]
Invariants revisited

We showed that
\[ s^2 = x^2 + y^2 + z^2 - c^2 t^2 = \bar{r}^2 - c^2 \bar{t}^2 \]
is a Lorentz invariant: it has the same value for a given measure of (where and when) in all frames of reference.

We will show that the same is true for the “dot product” of any two 4-vectors \((\vec{a}, a_0)\) and \((\vec{b}, b_0)\)
\[ a_\ell = \gamma(a_\ell' + \beta a_0') \quad b_\ell = \gamma(b_\ell' + \beta b_0') \]

\[ a_\perp = a_\perp' \quad b_\perp = b_\perp' \]

\[ a_0 = \gamma(a_0' + \beta a_\ell') \quad b_0 = \gamma(b_0' + \beta b_\ell') \]

\[
(\vec{a}, a_0) \cdot (\vec{b}, b_0) \equiv \vec{a} \cdot \vec{b} - a_0 b_0 \\
(\vec{a}, a_0) \cdot (\vec{b}, b_0) = a_\ell b_\ell + \vec{a}_\perp \cdot \vec{b}_\perp - a_0 b_0 \\
= \gamma^2 (a_\ell' + \beta a_0')(b_\ell' + \beta b_0') - \gamma^2 (a_0' + \beta a_\ell')(b_0' + \beta b_\ell') + \vec{a}_\perp \cdot \vec{b}_\perp' \\
= (a_\ell' b_\ell' - a_0' b_0') \gamma^2 (1 - \beta^2) + \vec{a}_\perp \cdot \vec{b}_\perp' \\
= (\vec{a}', a_0') \cdot (\vec{b}', b_0')
\]

So the scalar product of any two 4-vectors is a Lorentz invariant!
Energy and momentum

How to treat energy and momentum in special relativity? We must recover two cases:

**Classical limit:** \( v \ll c \), kinetic energy \( T = \frac{1}{2}mv^2 \)

**Relativistic limit:** \( v \to c \), no matter can travel \emph{faster} than the c, no matter how much kinetic energy it has.
Construct 4-vectors of energy, momentum

Einstein included two terms in the energy: the kinetic energy $T$ associated with motion, and the rest energy $mc^2$ associated with mass.

$$E \equiv T + mc^2$$

4-vector momentum $(\vec{p}, E)$

invariant

$$(\vec{p}, E)^2 = \vec{p}^2 c^2 - E^2$$

We can evaluate the invariant easily in the rest frame:

$p=0, T=0$, so

$$(\vec{p}, E)^2 = \vec{p}^2 c^2 - E^2 = -m^2 c^4$$
Now we can evaluate the velocity dependence of momentum, energy, kinetic energy, by making a Lorentz transform from the rest frame.

Rest frame: \( \bar{p}' = 0, \; T' = 0, \; E' = mc^2 \)

Boost to velocity \( \beta c \) in x direction:

\[
\begin{align*}
  p_x c &= \gamma (p_x' c + \beta E') = \beta \gamma mc^2 \\
  E &= \gamma (E' + \beta p_x' c) = \gamma mc^2 \\
  T &= E' - mc^2 = (\gamma - 1)mc^2
\end{align*}
\]
Now let’s recover the classical results:

\[ p = \beta \gamma mc \]

\[ \beta \ll 1, \quad \gamma \to 1, \quad p \approx m\beta c = mv \]

\[ T = (\gamma - 1)mc^2 \]

\[ \beta \ll 1, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}} \approx \frac{1}{1 - \frac{1}{2}\beta^2} \approx 1 + \frac{1}{2}\beta^2 \]

\[ T \approx \frac{1}{2}\beta^2mc^2 = \frac{1}{2}mv^2 \]

Indeed, we can define a relativistic transformation of mass:

\[ m_{lab} = \gamma m_{rest} \]
Suppose we accelerate a 10 ton spaceship to $v = 0.5c$. How much impulse do we deliver, how much work do we do?

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} = \frac{1}{\sqrt{1 - 0.5^2}} = 1.155$$

Impulse $\Delta p = m_{lab}v = (1.155 \cdot 10^5 \text{kg})(0.5 \cdot 3 \cdot 10^8 \text{ m/s}) = 1.73 \times 10^{19} \text{ kg m/s}$

Kinetic energy $T = (\gamma - 1)mc^2 = 0.155 \cdot (10^5 \text{ kg})(3 \cdot 10^8 \text{ m/s})^2 = 1.4 \times 10^{21} \text{ J}$

The output of a GW power plant for a year is $3 \times 10^{16} \text{ J}$!

This is comparable to the energy consumption of the entire Earth’s population for a year.
Suppose we keep accelerating?

As we try to increase $\beta$ further, gamma increases quadratically. The impulse and energy increase rapidly:

It would take an infinite amount of energy to reach $v=c$!
The work-energy theorem revisited

The work needed to accelerate a particle is just the change in kinetic energy:

\[ T_i = (\gamma_i - 1)mc^2 \]
\[ T_f = (\gamma_f - 1)mc^2 \]

\[ \text{Work done} = T_f - T_i = (\gamma_f - \gamma_i)mc^2 \]
Example: (5.37)

An electron is moving at \( v = 0.25c \).

If the speed is doubled, by what factor will \( T \) be increased?

\[
\gamma_i = \frac{1}{\sqrt{1 - \beta_i^2}} = 1.032
\]

\[
\gamma_f = \frac{1}{\sqrt{1 - \beta_f^2}} = 1.155
\]

\[
T_f - T_i = (\gamma_f - \gamma_i)mc^2 = 0.123 \cdot 511keV = 63keV
\]

Fractional increase = \( \frac{T_f - T_i}{T_i} = \frac{63}{527} = 0.12 \)
If the kinetic energy is increased by a factor of 100, by what factor will the speed be increased?

\[ T_f = 100T_i = 100 \cdot 1.032 \cdot mc^2 = 52.8\, MeV \]

\[ \gamma_f = \left(\frac{T_f}{mc^2} - 1\right) = 102 \]

\[ \beta_f = \sqrt{1 - \frac{1}{\gamma^2}} = 0.99995 \]

fractional change in \( \beta = \frac{\beta_f - \beta_i}{\beta_i} = 4.0 \)
Example of relativistic kinematics: Compton scattering of $\gamma$-rays

Momentum and energy are conserved. Both conditions are expressed by the 4-vector expression:

$$ q_i + p_i = q_f + p_f $$

where $q_i$, $q_f$ are the initial and final 4-momenta of the $\gamma$-ray, and $p_i$, $p_f$ are the initial and final 4-momenta of the electron.
\[ q_i = (E,0,0,E) \]
\[ p_i = (0,0,0,mc^2) \]
\[ q_f = (E'\cos \theta, E'\sin \theta,0,E') \]
\[ p_f = (p'\cos \Theta, -p'\sin \Theta, 0, \sqrt{p'^2 c^2 + m^2 c^4}) \]

Suppose we want an expression for \( E'(\theta) \).
Rewrite the conservation expression to isolate the 4-momentum we DON”T care about:
\[ q_i + p_i - q_f = p_f \]

Now form the invariant of each side:
\[ (q_i + p_i - q_f)^2 = p_f^2 = -m^2 c^4 \]
No terms in \( \Theta, p' \)!
\[(q_i + p_i - q_f)^2 = p_f^2 = -m^2 c^4\]
\[q_i^2 + p_i^2 + q_i^2 + 2q_i \cdot p_i - 2q_i \cdot q_f - 2p_i \cdot q_f\]
\[= 0 + (-m^2 c^4) + 0 + 2(0 - mc^2E) - 2(EE' \cos \theta - EE') - 2(-mc^2E')\]

So, solving for \(E'\),
\[
E' = \frac{mc^2E}{mc^2 + E(1 - \cos \theta)}
\]

This is the Klein-Nishina formula that accurately describes the scattering of \(\gamma\)-rays from electrons in matter.
In summary, we obtained the dependence of energy of the scattered $\gamma$-ray on the scattering angle, using the 4-vector notation for energy/momentum conservation, and the trick of isolating the 4-vector of the particle whose info we don’t know or need to calculate, so that when we take the square only its rest mass $m^2c^4$ carries into our calculation.
Now how do we transform between accelerated frames?

- Consider Newton’s first and second laws:
  \[ \vec{p} = m_i \vec{a} \]
  \[ \vec{F} = m_i \vec{a} \]
  $m_i$ is the measure of the inertia of an object – its resistance to a change in its state of motion.
  $m_i$ is the \textit{inertial mass} of an object.

- Now consider Newton’s Law of Gravitation:
  \[ \vec{F} = -G \frac{M m_g}{r^2} \hat{r} \]
  $m_g$ is a measure of the response of an object to gravitation.
  $M_g$ is the \textit{gravitational mass} of an object.
Rest mass of a decaying particle

Suppose that we have a particle of mass \( m \), traveling with momentum \( \vec{p}_0 \) in the lab. The particle decays into two new particles, masses \( m_1 \) and \( m_2 \). The decay products leave the decay with momenta \( \vec{p}_1, \vec{p}_2 \). Suppose that we want to obtain the mass \( m \) of the decaying particle, knowing only the masses and momenta of the decaying particles.
First we express the conservation of momentum and energy in 4-vector notation:

\[ p = p_1 + p_2 \]
\[ p = (\vec{p}_0, E_0) \]
\[ p_1 = (\vec{p}_1, E_1) \]
\[ p_2 = (\vec{p}_2, E_2) \]

Now we take the (Lorentz invariant) square of both sides:

\[ p^2 = -m_0^2 c^4 \]

\[ (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \]

\[ = (-m_1^2 c^4) + (-m_2^2 c^4) + 2(\vec{p}_1 \cdot \vec{p}_2 c^2 - E_1 E_2) \]

Putting it all together, in the limit \( |\vec{p}_1| c \approx E_1, \ |\vec{p}_2| c \approx E_2 \)

\[ -m_0^2 c^4 \approx -m_1^2 c^4 - m_2^2 c^4 - 2E_1 E_2 (1 - \cos \theta) \]

\[ m_0 \approx \sqrt{m_1 + m_2 + 2E_1 E_2 (1 - \cos \theta)} / c^2 \]
Equivalence Principle

Consider a man in an elevator, in two situations:

1) Elevator is in free-fall. Although the Earth is exerting gravitational pull, the elevator is accelerating so that the internal system appears inertial!

2) Elevator is accelerating upward. The man cannot tell the difference between gravity and a mechanical acceleration in deep space!

\[ m_i = m_g \]
But now for a paradox!

We shine a light into a window of the elevator, while the car is accelerating.

An inertial observer outside sees the light move on a straight-line trajectory. But the accelerated passenger feels either gravity or mechanical acceleration (he can’t tell which). She observes that the photons of light have mass-equivalent energy, and are therefore accelerated on a curved path!/
Einstein’s solution: space curves near a mass. Gravitational attraction is “straight” motion in a curved space.