The work-energy theorem revisited

The work needed to accelerate a particle is just the change in kinetic energy:

\[ T_i = (\gamma_i - 1)mc^2 \]
\[ T_f = (\gamma_f - 1)mc^2 \]

\[ \text{Work done} = T_f - T_i = (\gamma_f - \gamma_i)mc^2 \]
Example: (5.37)

An electron is moving at \( v = 0.25c \).

If the speed is doubled, by what factor will \( T \) be increased?

\[
\gamma_i = \frac{1}{\sqrt{1 - \beta_i^2}} = 1.032
\]
\[
\gamma_f = \frac{1}{\sqrt{1 - \beta_f^2}} = 1.155
\]

\[
T_f - T_i = (\gamma_f - \gamma_i)mc^2 = 0.123 \cdot 511\text{keV} = 63\text{keV}
\]

fractional increase \( = \frac{T_f - T_i}{T_i} = \frac{63}{527} = 0.12 \)
If the kinetic energy is increased by a factor of 100, by what factor will the speed be increased?

\[ T_f = 100T_i = 100 \cdot 1.032 \cdot mc^2 = 52.8 MeV \]

\[ \gamma_f = \left( \frac{T_f}{mc^2} - 1 \right) = 102 \]

\[ \beta_f = \sqrt{1 - \frac{1}{\gamma^2}} = 0.99995 \]

fractional change in \( \beta = \frac{(\beta_f - \beta_i)}{\beta_i} = 4.0 \)
Example of relativistic kinematics: Compton scattering of $\gamma$-rays

Momentum and energy are conserved. Both conditions are expressed by the 4-vector expression:

$$q_i + p_i = q_f + p_f$$

where $q_i$, $q_f$ are the initial and final 4-momenta of the $\gamma$-ray, and $p_i$, $p_f$ are the initial and final 4-momenta of the electron.
\[ q_i = (E,0,0,E) \]
\[ p_i = (0,0,0,mc^2) \]
\[ q_f = (E'\cos \theta, E'\sin \theta,0,E') \]
\[ p_f = (p'\cos \Theta,-p'\sin \Theta,0,\sqrt{p'^2 c^2 + m^2 c^4}) \]

Suppose we want an expression for \( E'(\theta) \).
Rewrite the conservation expression to isolate the 4-momentum we DON”T care about:
\[ q_i + p_i - q_f = p_f \]

Now form the invariant of each side:
\[ (q_i + p_i - q_f)^2 = p_f^2 = -m^2 c^4 \]  No terms in \( \Theta, p' \)!
\[
(q_i + p_i - q_f)^2 = p_f^2 = -m^2 c^4
\]
\[
q_i^2 + p_i^2 + q_i^2 + 2q_i \cdot p_i - 2q_i \cdot q_f - 2p_i \cdot q_f
\]
\[
= 0 + (-m^2 c^4) + 0 + 2(0 - mc^2 E) - 2(EE' \cos \theta - EE') - 2(-mc^2 E')
\]

So, solving for \( E' \),
\[
E' = \frac{mc^2 E}{mc^2 + E(1 - \cos \theta)}
\]

This is the Klein-Nishina formula that accurately describes the scattering of \( \gamma \)-rays from electrons in matter.
In summary, we obtained the dependence of energy of the scattered $\gamma$-ray on the scattering angle, using the 4-vector notation for energy/momentum conservation, and the trick of isolating the 4-vector of the particle whose info we don’t know or need to calculate, so that when we take the square only its rest mass $m^2c^4$ carries into our calculation.
Now how do we transform between accelerated frames?

• Consider Newton’s first and second laws:

\[ \vec{p} = m_i \vec{a} \]
\[ \vec{F} = m_i \vec{a} \]

\( m_i \) is the measure of the inertia of an object – its resistance to a change in its state of motion.

\( m_i \) is the \textit{inertial mass} of an object.

• Now consider Newton’s Law of Gravitation:

\[ \vec{F} = -G \frac{M m_g}{r^2} \hat{r} \]

\( m_g \) is a measure of the response of an object to gravitation.

\( M_g \) is the \textit{gravitational mass} of an object.
Suppose that we have a particle of mass $m$, traveling with momentum $p_0$ in the lab. The particle decays into two new particles, masses $m_1$ and $m_2$. The decay products leave the decay with momenta $\vec{p}_1, \vec{p}_2$. Suppose that we want to obtain the mass $m$ of the decaying particle, knowing only the masses and momenta of the decaying particles.
First we express the conservation of momentum and energy in 4-vector notation:

\[ p = p_1 + p_2 \]

\[ p = (\vec{p}_0, E_0) \]

\[ p_1 = (\vec{p}_1, E_1) \]

\[ p_2 = (\vec{p}_2, E_2) \]

Now we take the (Lorentz invariant) square of both sides:

\[ p^2 = -m_0^2 c^4 \]

\[ (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \]

\[ = (-m_1^2 c^4) + (-m_2^2 c^4) + 2(\vec{p}_1 \cdot \vec{p}_2 c^2 - E_1 E_2) \]

Putting it all together, in the limit \( |\vec{p}_1| c \approx E_1, \ |\vec{p}_2| c \approx E_2 \)

\[-m_0^2 c^4 \approx -m_1^2 c^4 - m_2^2 c^4 - 2E_1 E_2 (1 - \cos \theta) \]

\[ m_0 \approx \sqrt{m_1 + m_2 + 2E_1 E_2 (1 - \cos \theta) / c^2} \]
Equivalence Principle

Consider a man in an elevator, in two situations:

1) Elevator is in free-fall. Although the Earth is exerting gravitational pull, the elevator is accelerating so that the internal system appears *inertial*!

2) Elevator is accelerating upward. The man cannot tell the difference between gravity and a mechanical acceleration in deep space!

\[ m_i = m_g \]
But now for a paradox!

We shine a light into a window of the elevator, while the car is accelerating.

An inertial observer outside sees the light move on a straight-line trajectory. But the accelerated passenger feels either gravity or mechanical acceleration (he can’t tell which). She observes that the photons of light have mass-equivalent energy, and are therefore accelerated on a curved path!
Einstein’s solution: space curves near a mass. Gravitational attraction is “straight” motion in a curved space.
Rest mass of a decaying particle

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Putting it all together, in the limit \[ |\vec{p}_1| c \approx E_1, \quad |\vec{p}_2| c \approx E_2 \]

\[ -m_0^2 c^4 \approx -m_1^2 c^4 - m_2^2 c^4 - 2E_1 E_2 (1 - \cos \theta) \]

\[ m_0 \approx \sqrt{m_1 + m_2 + 2E_1 E_2 (1 - \cos \theta)} / c^2 \]