Example problem: Using potential to describe motion
Thornton & Marion problem 2.52

Problem: A particle of mass $m$ moves in a 1-D potential energy
$U(x) = U_0 \left[2 \left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^4 \right]$, where $U_0$ and $a$ are positive constants.

a) Find the force $F(x)$ which acts on the particle.

\[ F = -\frac{dU}{dx} = -4 \frac{U_0}{a} \left[ \frac{x}{a} - \left(\frac{x}{a}\right)^3 \right] \]

b) Sketch $U(x)$. Find the positions of stable and unstable equilibrium.

To find equilibrium, find values of $x$ where $F = 0$: $x = 0$, $\pm a$

Then test second derivative:

\[ \frac{d^2U}{dx^2} = -4 \frac{U_0}{a^2} \left[ -3 \left(\frac{x}{a}\right)^2 \right] \]

\[ = 4U_0 / a^2 > 0 \text{ at } x = 0 \Rightarrow \text{stable equilibrium} \]

\[ = -8U_0 / a^2 < 0 \text{ at } x = \pm a \Rightarrow \text{unstable equilibrium} \]

e) What is angular frequency of oscillations about the point of stable equilibrium?

Near $x = 0$, the lowest order term dominates, and the potential is approximately that of a simple harmonic oscillator:

$U \approx \frac{2U_0}{a^2} x^2$

$\omega = \frac{2}{a} \sqrt{\frac{U_0}{m}}$

d) What is the minimum speed the particle must have at the origin to escape to infinity?

It must pass over the maximum potential $U_0$ at $x = a$. Since the potential at the origin is zero, by the work energy theorem the minimum kinetic energy must by $U_0$. The minimum speed is then

\[ v_0 = \sqrt{\frac{2U_0}{m}} \]
e) At $t = 0$ the particle is at the origin and its velocity is positive and equal in magnitude to the escape speed. Find $x(t)$ and sketch the result.

We will use conservation of energy to obtain the kinetic energy $T(x)$, from it obtain $v(x)$, then integrate to obtain $t(x)$, and finally invert to obtain $x(t)$:

$$U_0 = U(x) + T(x)$$

$$u = x/a$$

$$T(u) = U_0 \left[1 - 2u^2 + u^4\right]$$

$$v(u) = \sqrt{\frac{2T(u)}{m}} = \sqrt{\frac{2U_0}{m} \left(1-u^2\right)}$$

$$t(u) = \int_0^u \frac{du}{v(u)} = \sqrt{\frac{ma^2}{8U_0}} \ln \left(\frac{1+u}{1-u}\right)$$

$$u(t) = \frac{e^{\sqrt{2\omega}t} - 1}{e^{\sqrt{2\omega}t} + 1} = \tanh \left(\frac{\alpha t}{\sqrt{2}}\right)$$

As expected, the particle asymptotically approaches the unstable equilibrium point, but never quite reaches it.