Problem 3.8 The Cycloidal Pendulum

A pendulum is supported at the cusp of a cycloid on a string of length \( \ell = 4a \). The coordinates of the plumb bob are parametrized by an angle \( \varphi \):
\[
x = a(\varphi - \sin \varphi) \\
y = a(1 - \cos \varphi)
\]

Prove that the pendulum is isochronous: the period does not depend upon the amplitude of oscillation.

We will develop an expression for the velocity as a function of \( \varphi \) using conservation of energy, then use that to calculate the period by integrating the motion.

\[
x' = a(1 - \cos \varphi) \\
y' = -a \sin \varphi
\]

\[
v = \frac{ds}{dt} = \sqrt{x'^2 + y'^2} \phi = a\sqrt{2}\sqrt{1 - \cos \varphi} \phi = \sqrt{2}x\phi
\]

Now conservation of energy connects the velocity \( v(\varphi) \) to the height of the motion \( y \). If \( y_0 \) is the highest point in an oscillation, then
\[
\frac{1}{2}mv^2 = mg(y_0 - y)
\]

We can now write an integral expression for the period:
\[
T = 4\int_0^{\varphi_0} \frac{ds}{v} = 4\int_0^{\varphi_0} \frac{\sqrt{x'^2 + y'^2}}{\sqrt{g(y_0 - y)}} d\varphi = 4\int_0^{\varphi_0} \frac{a}{\sqrt{g}} \frac{\sqrt{1 - \cos \varphi}}{\cos \varphi - \cos \varphi_0} d\varphi
\]

We can do the integral using the substitution
\[
u = \frac{\cos(\varphi/2)}{\cos(\varphi_0/2)} \\
du = -\frac{\sin(\varphi/2)d\varphi}{2\cos(\varphi_0/2)}
\]
\[
\cos \varphi = \cos^2(\varphi/2) - \sin^2(\varphi/2) = 2\cos^2(\varphi/2) - 1
\]
\[
\sqrt{1 - \cos \varphi} = \sqrt{2}\sin(\varphi/2)
\]
\[
\sqrt{\cos \varphi - \cos \varphi_0} = \sqrt{1 - u^2}\cos(\varphi_0/2)
\]
\[
T = 8\int_0^1 \frac{a}{g} \frac{du}{\sqrt{1 - u^2}} = \pi\sqrt{\frac{a}{g}}
\]

Thus the period does not depend upon the value of the amplitude \( \varphi_0 \) and the oscillator is isochronous. Note that this problem was first solved by Christian Huygens in 1673. Alas it is useless for timepieces, however, since friction on the contact surfaces of the cycloid produces a larger error than is introduced by non-isochronicity.

You can see a nice animation of the problem at
http://mathworld.wolfram.com/TautochroneProblem.html