**Physics 302**
**Exam 1**

1. An airplane is flying due west at a location $0^o$ longitude, $\alpha$ = $30^o$N latitude, with a ground speed of $v_g$ = 300 m/s.

   a. Calculate the position, velocity, and acceleration of the plane in an INERTIAL Cartesian coordinate frame with its origin at the center of the Earth. Neglect effects of the Earth's orbit around the Sun, the orbit of the Sun in the Milky Way, etc. Specify your INERTIAL frame clearly, with a sketch.

Define an inertial frame with its origin at the center of the Earth, $\hat{z}$ pointing through the N pole, and $\hat{x}$ oriented through $0^o$ longitude at time t=0 (when the plane has that longitude). The Earth rotates with an angular velocity

$\Omega = \dfrac{2\pi}{24hr}$, with rotation from W to E (clockwise viewed from S pole). That is the positive direction for angular velocity, consistent with the above choice of (right-handed) coord frame.

The net angular velocity of the plane in the inertial frame is

$$\omega = \Omega - \frac{v_g}{R\cos\alpha}$$

The position is then
$$x = R\cos\alpha\cos\omega t$$
$$y = R\cos\alpha\sin\omega t$$
$$z = R\sin\alpha$$

The velocity is
$$\dot{x} = -R\omega\cos\alpha\sin\omega t$$
$$\dot{y} = R\omega\cos\alpha\cos\omega t$$
$$\dot{z} = 0$$

The acceleration is
$$\ddot{x} = -R\omega^2\cos\alpha\cos\omega t$$
$$\ddot{y} = -R\omega^2\cos\alpha\sin\omega t$$
$$\ddot{z} = 0$$

   b. What is the clock time from sunrise to sunrise on board the plane if it maintains this motion long enough?

The time for sunrise to sunrise for the plane is $T_p = \dfrac{2\pi}{\omega}$

The duration of an Earth day is $T_e = \dfrac{2\pi}{\Omega}$

The time in days for a sunrise cycle on the plane is

$$\frac{T_p}{T_e} = \frac{\Omega}{\Omega - \dfrac{v_g}{R\cos\alpha}} = \frac{1}{1 - \dfrac{v_g}{\Omega R\cos\alpha}}$$

For the parameters of this problem, this is ~3.5 days! That is because the plane is flying a little *faster* than the shadow of sunrise is moving on the surface of the Earth at that latitude.

2. The Morse potential energy $U = U_0\left(1-e^{-a(r-r_0)}\right)^2$ provides a good approximation to the bound states of some binary molecules.

a. Calculate the radius $r_e$ of the equilibrium separation.

We want $\frac{\partial U}{\partial r}(r_e) = 0 = U_0\left[2ae^{-a(r_e-r_0)} - 2ae^{-2a(r_e-r_0)}\right]$

$r_e = r_0$

b. Assuming the molecule is a bound state of two identical atoms, calculate the resonant frequency of small-amplitude oscillations. (Remember that both atoms move!)

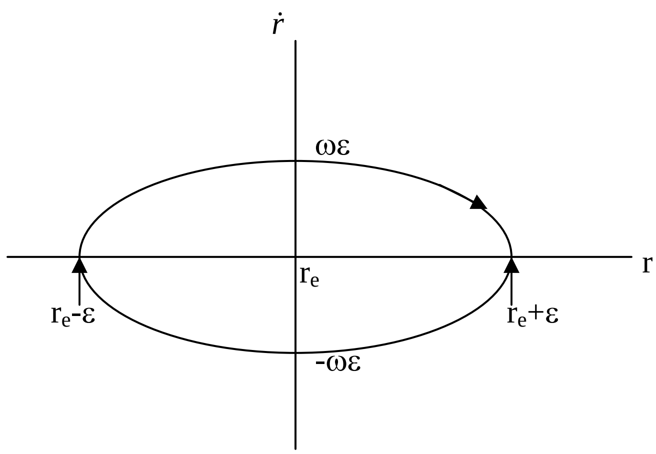
Since the two atoms are identical, they will move symmetrically ('breathing mode') in a radial oscillation. The total kinetic energy is $E = 2\left(\frac{1}{2}m\dot{x}^2\right) + U(x) \cong U(x_e) + m\dot{r}^2 + \frac{d^2U}{dr^2}(r_e)\frac{r^2}{2}$

$$\frac{d^2U}{dr^2}(r_e) = U_0\left[-2a^2 + 4a^2\right] = 2U_0a^2$$

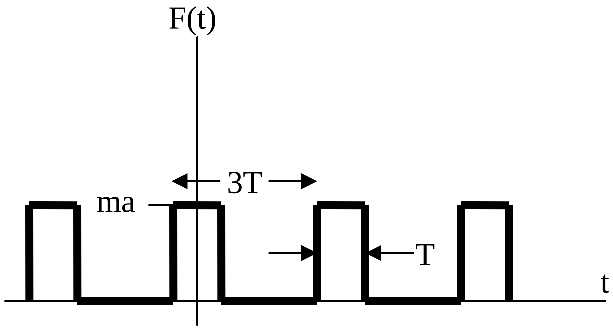
The oscillation takes the form $r = r_1\cos\omega t$ so the angular frequency is

$$\omega = a\sqrt{\frac{U_0}{m}}$$

c. Assume that two identical atoms have a separation $r(0) = r_e + \varepsilon$, where $\varepsilon << r_e$. Sketch the trajectory of the molecule in a phase diagram in the variables $(r, \dot{r})$. Indicate clearly on your sketch the scale on each axis and the direction in which the system evolves as time goes forward.



3. Consider a periodic driving force F(t) as shown below:




Calculate the first three harmonic terms in the Fourier decomposition of this force.

$$\frac{F(t)}{m} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right)$$

We calculate the constant term from the time average of the force:

$$a_0 = 2\left\langle \frac{F}{m} \right\rangle = \frac{2}{3T}\left[a \cdot T + 0 \cdot 2T\right] = \frac{2a}{3}$$

The force is an even function of time, *F(t) = F(-t)*, so all $b_n$=0.

We now choose the fundamental frequency for the decomposition to be the frequency with which the force repeats itself: $\omega = \frac{2\pi}{3T}$

The $a_n$ terms are calculated using Fourier's trick:

$$a_n = \frac{2}{3T}\int_0^{3T} \frac{F(t)}{m} \cos n\omega t dt = \frac{2}{3T}\frac{2a}{n\omega}\sin\frac{n\omega T}{2} = \frac{2a}{n\pi}\sin\frac{\pi n}{3}$$

The first three harmonics are thus

$$a_0 = \frac{2a}{3}$$

$$a_1 = \frac{\sqrt{3}a}{\pi}$$

$$a_2 = \frac{\sqrt{3}a}{2\pi}$$

4. A straight hole is drilled through the spherical Earth along a chord having total length L < 2R. A ball is dropped into the hole at one end. It is free to move along the hole (no friction or air drag) but constrained to move only along the chord. Assume uniform density in the Earth. Calculate the distance x(t) (with respect to the point into which it is dropped) for the resulting motion. Describe it in words.

The gravitational force at a radius r within the Earth is obtained using the shell theorem:

$$\vec{F} = -G\frac{Mmr}{R_e^3}\hat{r}$$

Now consider a chord of total length L through the Earth. At a location a distance x from the midpoint of the chord, the component of the gravity force along the direction of the chord is

$$F_x = F\frac{x}{r} = -G\frac{Mm}{R^3}x$$

The motion of the ball along the chord is then simple harmonic motion:

$$m\ddot{x} = -G\frac{Mm}{R^3}x$$

The angular frequency of oscillation is $\omega = \sqrt{\frac{GM}{R^3}}$

Note that the frequency of oscillation is independent of how shallow or deep the chord is. Taking the parameters for the Earth, we obtain a period of oscillation:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\left(6.4\cdot 10^6 m\right)^3}{\left(6.7\cdot 10^{-11} N/m^2/kg^2\right)\left(6\cdot 10^{24} kg\right)}} = 1.4 days$$

So the ball will perform simple harmonic motion along the length of the chord, just reaching the far end and then returning, with a period of 2 days. The period of the motion is independent of the overall depth of the chord.