Physics 302
Exam 2

6. Find the curve that passes through endpoints (0,0) and (1,1) and minimizes

$$J = \int_0^1 \left[y'^2 - y^2 \right] dx$$

Solution: Since the integrand *f* does not depend upon the variable of integration *x*, we will use the second form of Euler's equation:

$$f - y' \frac{\partial f}{\partial y'} = c^2 \qquad \left(y'^2 - y^2 \right) - y'(2y') = c^2 \qquad y^2 + y'^2 = c^2$$

$$y = \sin x$$

Note that we chose the sin *x* solution rather than the cos *x* solution, and the value *c=1*, in order to satisfy the end point conditions.

$$J_{\min} = \int_0^1 \left[\cos^2 x - \sin^2 x \right] dx = \int_0^1 \cos 2x dx = \left. \frac{\sin 2x}{2} \right|_0^1 = \frac{\sin 2}{2} = 0.455$$

For a straight line, $y=x$ and

$$J_{straight} = \int_0^1 \left[1 - x^2 \right] dx = \left[x - \frac{x^3}{3} \right]_0^1 = \frac{2}{3}$$

7. A bead of mass *m* is constrained to move without friction on a circular wire hoop of radius *R*, which itself is rotating about a vertical diameter with constant angular velocity ω. Find the equilibrium position of the bead. Now suppose that it is displaced a small distance from its equilibrium position. Set up Lagrange's equations for its motion on the hoop, and calculate the angular frequency of small-angle oscillations. Find and interpret physically a critical angular velocity ω_c that divides the bead's motion into two distinct types. Construct phase diagrams for the two cases $\omega>\omega_c$, $\omega<\omega_c$.

Solution: Define the location of the bead by its angle θ with respect to the bottom of the hoop. Equilibrium corresponds to the condition that the force in the direction in which the wire is free to move is just what is needed to produce the circular motion of the bead:

$$g\sin\theta = \omega^2 (R\sin\theta)\cos\theta$$

$$\cos\theta = \frac{g}{R\omega^2}$$

Now displace the bead from equilibrium by an angle $d\theta$:

$$T = \frac{1}{2}m\omega^2 (R\sin\theta)^2 + \frac{1}{2}mR^2\dot{\theta}^2$$

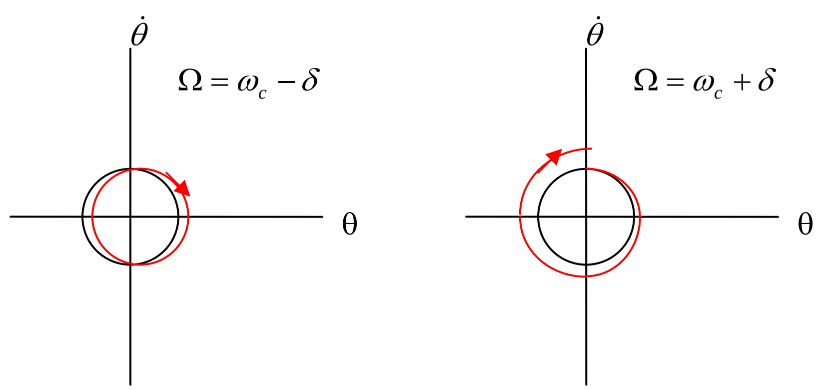
$$U = mgR(1-\cos\theta)$$

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = m\omega^2 R^2 \sin\theta\cos\theta - mgR\sin\theta - mR^2\ddot{\theta} = 0$$

$$\ddot{\theta} \cong \left(\omega^2 - \frac{g}{R}\right)\theta$$

$$\theta = \theta_0 e^{i\Omega t} \qquad\qquad \Omega = \sqrt{\omega^2 - \frac{g}{R}}$$

The motion changes character when the radicand changes sign; the exponent in the θ motion changes from imaginary to real, and the motion changes from harmonic oscillation about stable equilibrium to exponential divergence from an unstable equilibrium. The critical value of ω is $\omega_c = \sqrt{g/R}$. The phase diagrams are:



8. Some folks have proposed disposing of nuclear waste by either carrying it out of the solar system or by crashing it into the sun. Assume that no 'sling-shot' scenarios are permitted and that thrusts occur only in the orbital plane. Calculate the minimum amount of impulse Δv that is required in each case.
G = 6.67 x 10^{-11} m^3/s^2kg, M_{sun} = 2 x 10^{30} kg, $R_{earth-sun}$= 1.5x 10^{11} m, R_{sun} = 7 x 10^8 m, M_{earth} = 6 x 10^{24} kg.

Solution:
Escape the solar system:
The minimum escape velocity Δv_e to leave the solar system from Earth orbit is obtained by energy conservation:

$$\frac{GM_s}{R_{es}^2} = \frac{v_o^2}{R_{es}}$$

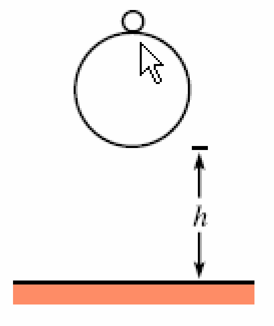
$$-\frac{GM_s}{R_{es}} + \frac{1}{2}(v_0 + \Delta v_e)^2 = 0$$

$$-v_0^2 + (2v_0 + \Delta v_e)\Delta v_e = 0$$

$$\Delta v_e = \left(-1 + \sqrt{2}\right)v_0$$

To crash into the sun, we simply require to stop it from orbiting the sun; i.e. we give it a boost $\Delta v_s = -v_0$
So it takes less boost velocity to eject from solar orbit than to crash into the sun.

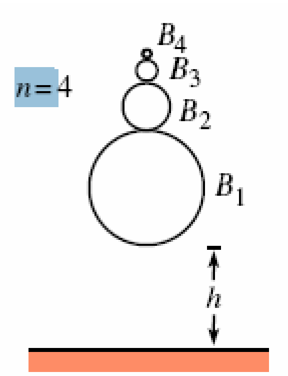
9. a) A tennis ball of mass m_2 sits atop a basketball of mass $m_1 >> m_2$. The balls are released from rest when the bottom of the basketball is a height *h* above a horizontal surface. To what height does the tennis ball bounce? Ignore wind velocity, assume that the ball diameters are negligible, assume that the balls bounce elastically.



For simplicity, assume that the balls are separated by a very small distance, so that the relevant bounces happen a short time apart. This assumption isn't necessary, but it makes for a slightly cleaner solution.

Just before the basketball hits the ground, both balls are moving downward with speed $v_0 = \sqrt{2gh}$. When the basketball bounces off the ground, it moves upward with speed v_0, while the tennis ball still moves downward with speed v_0. The relative speed is therefore $2v_0$. The center-of-mass frame is approximately that of the basketball ($m_1>>m_2$) so after the basketball bounces the center-of-mass frame is moving upwards with velocity $V=v_0$. In that frame the back-to-back scattering simply reverses the signs of the two momentum vectors. The tennis ball thus has an upwards lab-frame velocity of $2v_0+V=3v_0$. By conservation of energy, it will therefore rise to a height of $h_f = h + \frac{(3v_0)^2}{2g} = 10h$

b) Now consider n balls B_1, B_2, B_3, … B_n, having masses $m_1>>m_2>>\ldots>>m_n$, sitting in a vertical stack. The stack is released from rest when the bottom of ball B_1 is *h* above the ground. In terms of n, to what height does the top ball bounce? If *h* = 1 m, what is the minimum number of balls required for the top one to reach escape velocity from Earth?



Solution: Just before ball B_1 the ground, all of the balls are moving downward with speed $v_0 = \sqrt{2gh}$. We will inductively determine the speed of each ball after it bounces off the one below it. If ball B_i achieves a speed of v_i after bouncing off ball B_{i-1}, then what is the speed of ball B_{i+1} after it bounces off ball B_i? The relative speed of balls B_{i+1} and B_i (right before they bounce) is $v + v_i$. This is also the relative speed after they bounce. Since ball B_i is still moving upwards at essentially speed v_i, the final upward speed of ball B_{i+1} is therefore $(v + v_i) + v_i$. Thus,

$$B_{i+1} = 2v_i + v_0$$
$$B_3 = 7v_0,\ B_4 = 15v_0,\ \text{etc.}$$

From conservation of energy, ball B_n will bounce to a height of

$$h_n = h + \frac{\left((2^n - 1)v_0\right)^2}{2g} = h\left[1 + (2^n - 1)^2\right]$$

If h is 1 meter, and we want this height to equal 1000 meters, then (assuming the rest of the balls are not very large) we need $(2^n - 1) > 1000$. Six balls are sufficient.

Escape velocity from the earth is $v_e = \sqrt{2gR_e} = 11km/s$. That speed is reached when

$$(2^n - 1)\sqrt{2gh} > \sqrt{2gR_e}$$
$$n > \log_2\left(1 + \sqrt{R/h}\right) \sim 12$$

So a stack of 12 balls should suffice to send the top one to the moon!