Physics 302
Pretest Chapter 6

1. Consider two points (x,y,z) = (1,1,1) and (√3,0,0) located on the surface of a sphere of radius √3. Use the technique of Euler's equations to derive the geodesic curve on the surface of a sphere that connects the two points on the surface with minimum path length. What is the path length? Show all steps! Draw a figure and label variables. Interpret your result.

Answer: The path length in spherical polar coordinates is

$$J = \int_{\vec{r}_1}^{\vec{r}_2} R \underbrace{\sqrt{1+\sin^2\theta\phi'^2}}_{f(\theta,\phi')} d\theta$$

$$\frac{\partial f}{\partial \phi} - \frac{d}{d\theta}\frac{\partial f}{\partial \phi'} = -\frac{d}{d\theta}\left(\frac{\phi'}{\sqrt{1+\sin^2\theta\phi'^2}}\right) = 0$$

$$\phi' = \frac{c}{\sqrt{c^2 - \sin^2\theta}}$$

The conversion of this expression into the equation of a plane through the center of the sphere is given in the text. The intersection of such a plane is a segment of a great circle, an arc on the spherical surface that is a segment of a circle with center at the center of the sphere.

The geodesic between the given points is an arc between an axis and a neighboring body-centered quadrant. If we continue around the same great circle in the opposite direction, we would go to (0,1,1), which is perpendicular to (1,1,1) along this geodesic. Thus our arc is one-eighth of a full circle, and has path length

$$s = \frac{\sqrt{3}\pi}{4}$$

2. Consider a piece of string of length L, mass/length λ. Find the equation of the catenary curve that the string will form if it hangs freely with the ends fixed to two points (x_1,y_1) and (x_2,y_2) that are separated by a distance d<L.
You will receive full credit if you express the answer correctly as a simple integral, 5 pts bonus if you solve the integral!

Solution:

We want to minimize the total potential energy of the string:

$$dU = gy(\lambda ds)$$

$$U = \int_{\vec{r}_1}^{\vec{r}_2} \underbrace{g\lambda y\sqrt{1+x'^2}}_{f(x',y)}\,dy$$

Note: we choose y to be the dependent variable instead of x, so that the dependent variable does not appear in the functional.

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = -\frac{d}{dx}\left(\frac{yx'}{\sqrt{1+x'^2}}\right) = 0$$

$$yx' = C\sqrt{1+x'^2}$$

$$x' = \frac{\pm C}{\sqrt{y^2 - C^2}}$$

$$x(y) = \int_{y_1}^{y_2} \frac{C|dy|}{\sqrt{y^2 - C^2}} + x_0$$

$$x - x_0 = \cosh^{-1}(y - y_0)$$

$$y - y_0 = \cosh(x - x_0)$$

The constants of integration, x_0 and y_0, are chosen so that the catenary passes through the designated end points.