Physics 302
Chapter 7 Pretest

A pendulum of mass m is suspended from a massless string of length *l*, which in turn is fixed at its other end to a pivot on a cart of mass M that is free to slide without friction along a horizontal track.

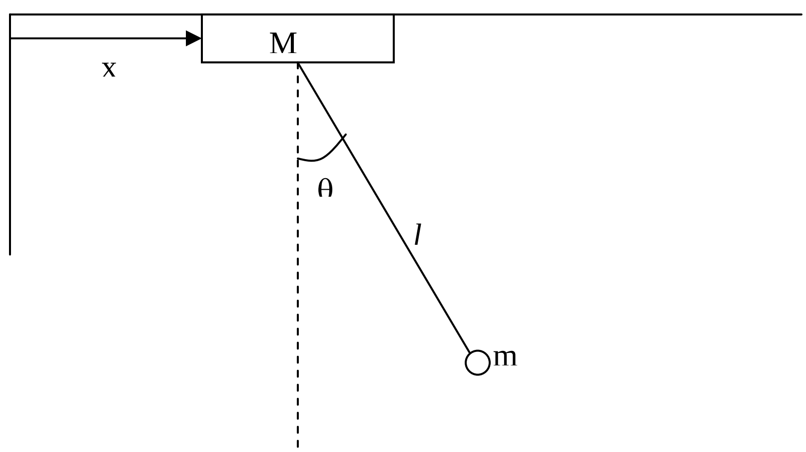
a) Make a sketch, label suitable coordinates, and set up Lagrange's equations.

b) Show that there is an ignorable coordinate. Interpret motion in this coordinate in terms of conservation principles.

c) Solve for the motion in the small-amplitude limit. Show that this reduces to the solution for a pendulum about a fixed pivot in an appropriate limit.

d) Plot the motion as a trajectory in an appropriate phase space. Label limits of the motion in terms of the parameters of your solution. Show the direction of time progression on the trajectory.

Solution:



The variable *x* is the (horizontal) location of the car. The variable θ is the angle of the string to the vertical.

The potential and kinetic energies are

$$U = mgl(1-\cos\theta)$$

$$T = \frac{1}{2}m\left(l\dot{\theta}+\dot{x}\right)^2 + \frac{1}{2}M\dot{x}^2$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\frac{d}{dt}\left[m\left(l\dot{\theta}+\dot{x}\right)+M\dot{x}\right] = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -mgl\sin\theta - ml\left(l\ddot{\theta}+\ddot{x}\right) = 0$$

The Lagrangian does not contain any term in *x*, only in $\dot{x}$, so *x* is an ignorable coordinate. The solution of the Lagrange's equation in *x* just yields conservation of linear momentum in the horizontal direction (in which there is no external force):

$$\dot{x} = -\frac{ml\dot{\theta}}{M+m}$$

Substituting this in Lagrange's equation in θ yields

$$-mgl\sin\theta - ml^2\left(1-\frac{m}{M+m}\right)\ddot{\theta} = 0$$

In the small angle limit this is just simple harmonic motion with a frequency

$$\omega = \sqrt{\frac{g}{l}}\sqrt{\frac{(M+m)}{M}}$$

In the limit *M*>>*m*, this reduces to the familiar frequency of a simple pendulum oscillating about a fixed center.